Discrete Math

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Set Theory

Unordered, well defined, collection of distinct objects.

Ex → $A = \{1, 3, 5 \ldots\}$

$X = \{n \mid n \in \text{Natural Numbers}\}$

$x \in A$ means element x is member of A

Cardinality of Set : No. of elements present in a set, denoted by $|A|$.

Ex → $A = \{0, 2, 4, 6\}$, $|A| = 4$

Representation of Set

→ Tabular : List all members.

Ex → $A = \{a, e, i, o, u\}$

→ Set Builder: Specify properties which a element must satisfy to be part of set.

Ex → $A = \{x \mid x \text{ is vowel}\}$

Classification

→ Finite set : Ex - $A = \{1, 2, 3\}$

→ Infinite set : Ex - A set of natural no.s

→ Null set / Empty set : Cardinality is 0. Denoted by ϕ or $\{\}$ (phi)

→ Universal set : All sets under investigation are subset of this set. Denoted by U

Subset

If every element of set A are in set B then A is subset of B. $(A \subseteq B)$

$\forall x (x \in A \rightarrow x \in B)$

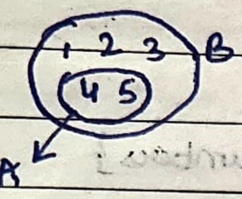
→ $\phi \subseteq A$ Empty set is subset of every set.

→ $A \subseteq U$ Every set is " " Universal ".

→ $A \subseteq A$ " " " " itself.

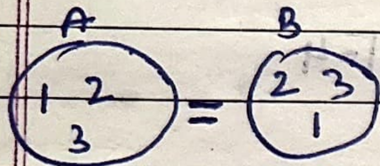
Proper Subset

If A is subset of B & $A \neq B$. Denoted by $A \subset B$



Equality of Sets

If $A \subseteq B$ & $B \subseteq A$ the $A = B$



Power Set

Set of all subset is called Power set.

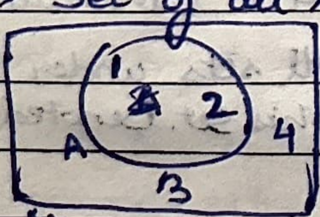
$A = \{1, 2, 3\}$

$P(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{1,3\}, \{1,2,3\}\}$

Cardinality of $P(A) \Rightarrow |P(A)| = 2^n$

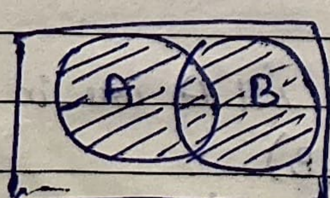
Operations on sets

→ **Complement** → Set of all x, s.t $x \notin A$ & $x \in U$



$A' = \{3, 4\}$

→ **Union** →

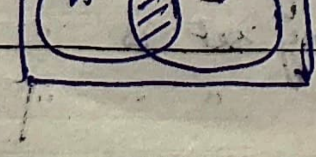


$A = \{1, 2\}$

$B = \{2, 3\}$

$A \cup B = \{1, 2, 3\}$

→ **Intersection** →



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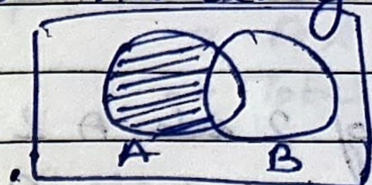
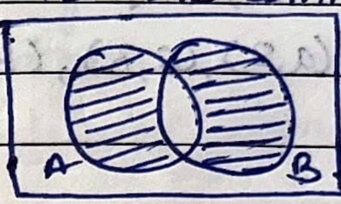
→ <u>Set Difference</u> → A − B, Give elements of A that do not belong to B.



→ <u>Symmetric Difference</u> → $A \oplus B$, Give element of both A & B but not the common elements.



<u>Rules</u>

- <u>Idempotent Law</u> : $A \cup A = A$
  $A \cap A = A$

Associative Law : $(A \cup B) \cup C = A \cup (B \cup C)$
$(A \cap B) \cap C = A \cap (B \cap C)$

Commutative Law : $A \cup B = B \cup A$
$A \cap B = B \cap B$

Distributive Law : $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

De-Morgans Law : $(A \cup B)' = A' \cap B'$
$(A \cap B)' = A' \cup B'$

Identity Law : $A \cup \phi = A$
$A \cap \phi = \phi$
$A \cup U = U$
$A \cap U = A$

Complement Law : $A \cup A' = U$
$A \cap A' = \phi$
$U' = \phi$
$\phi' = U$

Involution Law : $(A')' = A$

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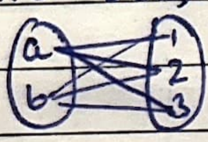
Relations

Cartesian Product

Cartesian Product of 2 sets A & B is set of all ordered pairs where a is in A & b is in B.
It is denoted by $A \times B$.

Ex → $A = \{a, b\}$ $B = \{1, 2, 3\}$

$A \times B = \{(a,1), (a,2), (a,3), (b,1), (b,2), (b,3)\}$



Note → • Commutative law doesn't apply $A \times B \,!= B \times A$
• Cardinality → $|A| = m$, $|B| = n$ then $|A \times B| = m \cdot n$

Relation

Let A & B are sets then every possible subset of '$A \times B$' is relation from A to B.
Total relations possible → $|A| = m$, $|B| = n$, $|A \times B| = m \cdot n$
Total Relations $= 2^{m \cdot n}$

Inverse of Relation

If R is relation from A to B then R^{-1} will be relation from B to A.

$A = \{a, b\}$ $B = \{1, 2\}$

$A \times B = \{(a,1), (a,2), (b,1), (b,2)\}$

$R = \{(a,1), (b,2)\}$

$R^{-1} = \{(1,a), (2,b)\}$

Diagonal Relation

$R = \{(x,x) \mid \forall x \in A\}$

	1	2	3
1	X		
2		X	
3			X

Only $\{(1,1), (2,2), (3,3)\}$ should be present.

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Types of Relation

Assumptions → set A with n element
→ A × A will thereby have n^2 elements
→ Total Relations possible $2^{n \times n}$

→ **Reflexive Relation**

If $\forall x \in A$ then $(x, x) \in R$

Give all diagonal elements, rest dont care can be or not be there.

Ex→ $A = \{1, 2, 3\}$ then $\{(1,1), (2,2), (3,3)\}$ ✓
$\{(1,1), (2,2), (3,3), (1,2)\}$ ✓
$\{(1,1) (2,2), (1,2)\}$ X → missing (3,3)

→ **Irreflexive Relation**

If $\forall x \in A$ then $(x, x) \notin R$

Anything but diagonal elements.

→ **Symmetric Relation**

If $\forall a, b \in A$
$(a, b) \in R$

then $(b, a) \in R$

→ **Anti Symmetric Relation**

If $\forall a, b \in A$
(aRb) & (bRa) then
$a = b$

(If 2 elements are related in both directions then they must be same element)

→ **Transitive Relation**

If $(a,b) \in R$, $(b,c) \in R$

then $(a, c) \in R$

→ **Equivalence Relation**

If Relation is Reflexive, Symmetric & Transitive

Ex→ ~~Ea~~ R: (a, b) iff $(a+b)$ is even over set of Integers

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→ Partial Order Relation

If Relation is Reflexive, ~~Sy~~ Anti-Symmetric & Transitive.

Ex→ R : (a,b) iff $b/a \in Z$ (Perfect Integer i.e no frac.)

POSET & ~~Hasse Diagram~~ Lattices

POSET (Partial Ordering Set):

A set with Partial ordering relation R. Denoted by [A, R].

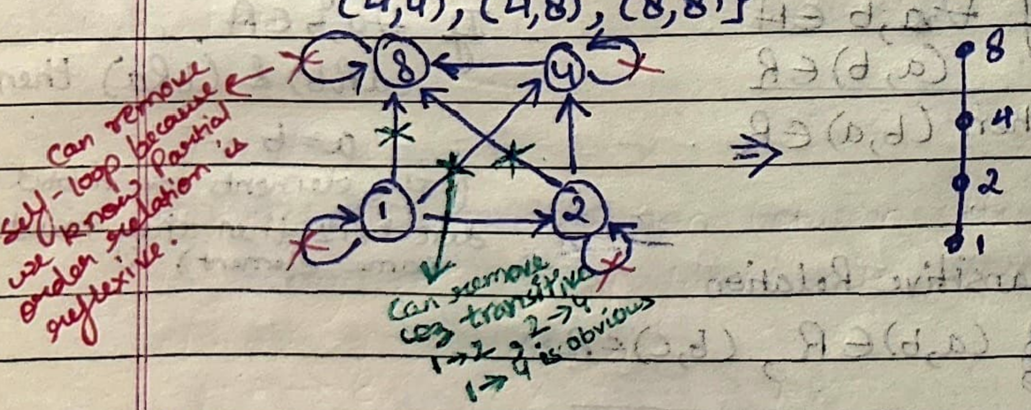
Ex → [A, /]

Hasse Diagram

Graphical representation of Partial order relation

Ex→ Lets say we have a POSET

$R = \{(1,1), (1,2), (1,4), (1,8), (2,2), (2,4), (2,8), (4,4), (4,8), (8,8)\}$



Steps → ⊙ Draw vertex for each element in set.

⊙ If (a,b) ∈ R then draw edge from a to b.

⊙ Remove all Reflexive & Transitive edges

⊙ Remove direction of edges & ~~increase~~ arrange in increasing order of heights.

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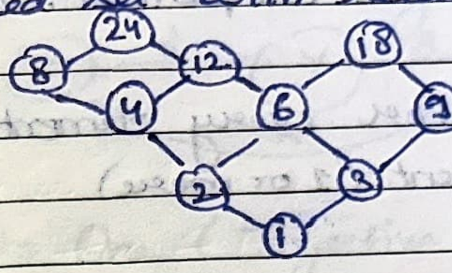
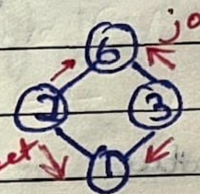
Lattice Ex 2 → Let $A = (1,2,3,4,6,8,9,12,18,24)$ be ordered set with relation "x divides y".



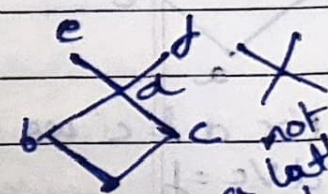
Lattice

A hasse diagram / Partial order relation is Lattice if their exists a join & meet for every pair of elements.

Ex 1 →

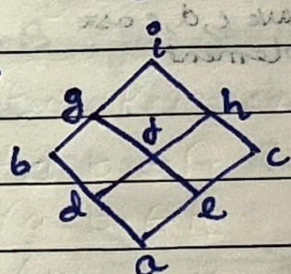


meet of 2 & 3 is 1.
join of 2 & 3 is 6.

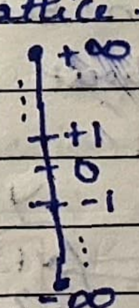


not lattice as join of e & f does not exist.

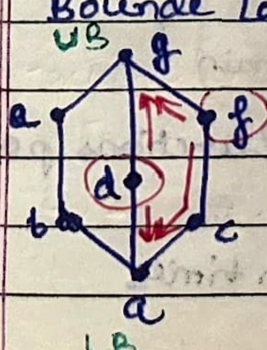
Ex 2 →



Unbounded Lattice → If lattice has infinite elements.



Bounde Lattice → Finite elements



$a \vee a^c = UB$ (upper bound)

$a \wedge a^c = LB$

d & f are complement as $d \vee f = g$ (UB)

$d \wedge f = a$ (LB)

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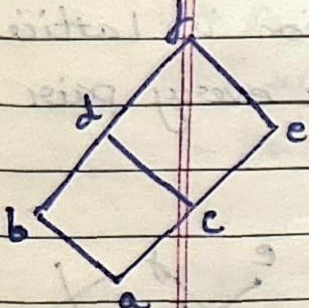
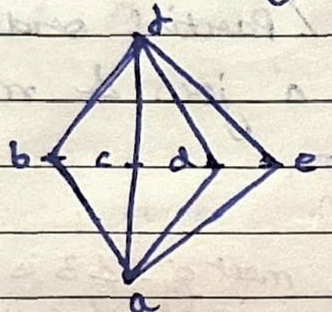
Distributive Lattice → A Lattice, where, for every element there exist at most 1 complement. (0 or 1)

Complement Lattice → For every element their exist at least 1 complement. (1 or more)

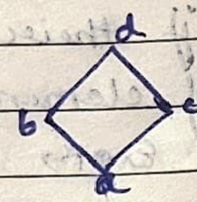
Boolean Algebra → A lattice is B.A if for every element there exists only 1 complement.



For d & c we get
$d \vee c = f$
$d \wedge c = c$ (not a LB)
so d & c aren't complement
Distributive Lattice



Complement Lattice
For b we have c, d, e as complement

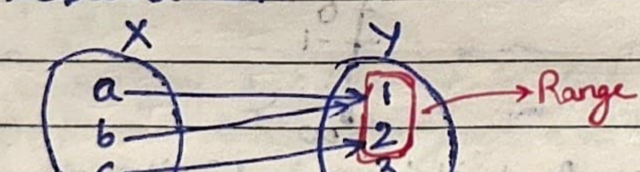


Complement Lattice
Distributive Lattice
Boolean Algebra

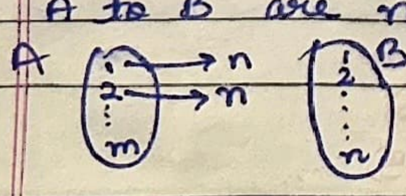
Functions

A function is a relation between sets that associates to every element of first set exactly one element of second set.

$f: A \to B$



Note → If $|A| = m$ & $|B| = n$, then no. of functions possible from A to B are n^m



$n \times n \times \ldots \, m$ times

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Function Composition → Process of combining 2 or more functions. $h(x) = g(f(x))$

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[rr, bend right, "h"'] & y \arrow[r, "g"] & z \end{tikzcd}$$

$z = g(y) = g(f(x))$

→ One to One (Injective Function)

In a function $F: A \to B$, if for every element of A has distinct image in B. $|A| \leq |B|$

→ For unique mapping of B.

→ Not necessarily means that Range should be equal to Co-Domain

No. of functions possible = ${}^{n}P_{m}$

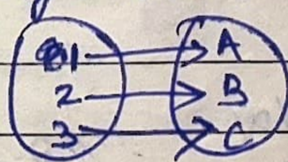
→ Onto (Surjective Function)

$F: A \to B$, for every element of B, there is atleast 1 value in A where $f(a) = b$. $|A| >= |B|$

Or Codomain = Range.

→ Bijective Function

Each element of set 1 is exactly paired with 1 element of other set. $|A| = |B|$



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Inverse of a Function

Can only be applied to Bijective function.

$f(x) = y$ Then $f^{-1}(y) = x$

Ex → $f(x) = 5x - 7$

$$f^{-1}(y) = \frac{(y+7)}{5}$$

Theory of Logic

Proposition

Declarative sentence (a sentence that declares fact) that is either true or false but not both.

Premise → Always considered to be true, a statement that provides reason or support for conclusion.

Operators

→ Negation ($\sim$)

p	$\sim p$
F	T
T	F

→ Conjunction ($\wedge$)

p	q	$p \wedge q$
F	F	F
F	T	F
T	F	F
T	T	T

Conjunctive Syllogism

$P_1 \rightarrow \sim(p \wedge q)$

$P_2 \rightarrow p$

$Q \rightarrow \sim q$

→ Disjunction ($\vee$)

p	q	$p \vee q$
F	F	F
F	T	T
T	F	T
T	T	T

Disjunctive Syllogism

$P_1 \rightarrow (p \vee q)$

$P_2 \rightarrow \sim p$

q

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→ Implication ($\rightarrow$) : If p then q ($p \rightarrow q$)

p	q	$p \rightarrow q$
F	F	T
F	T	T
T	F	F
T	T	T

Note 1 → $p \rightarrow q$ Implication

$q \rightarrow p$ Converse

$\sim p \rightarrow \sim q$ Inverse

$\sim q \rightarrow \sim p$ Contra positive

Note 2 → ⊙ $p \rightarrow q = \sim q \rightarrow \sim p$

⊙ $p \rightarrow q = \sim p \vee q$

Ques → Express Converse, Inverse & Contrapositive of the following statement "If $x+5=8$ then $x=3$"

Converse = If $x=3$ then $x+5=8$

Inverse = If $x+5 \neq 8$ then $x \neq 3$

Contra-Positive = If $x \neq 3$ then $x+5 \neq 8$

Modus Ponen

P_1 : $p \rightarrow q$

P_2 : p

Q : q

proof

p	q	$p \rightarrow q$
F	F	T
F	T	T
T	F	F
T	T	T

p $p \rightarrow q$ q

Modus Tollens

P_1 : $p \rightarrow q$

P_2 : $\sim q$

Q : $\sim p$

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→ Bi-Conditional $p \leftrightarrow q = (p \to q) \wedge (q \to p)$
$(\leftrightarrow)$

If and Only If

p	q	$p \leftrightarrow q$
F	F	T
F	T	F
T	F	F
T	T	T

Types of Cases

→ Tautology / Valid : Always true

p	$\sim p$	$p \vee \sim p$
F	T	T
T	F	T

→ Cotradiction : Alway False

→ Contingency: Neither tautology or contradiction

→ Satisfiable: Atleast 1 true

First Order Predicate Logic

We use FOPL to deal with shortcomings of Propositional logic. In this we understand new approach of subject & predicate to extract more information from a statement.

→ Predicate → Functions that return True or False based on properties of object.

→ Quantifiers → Symbols that specify quantity of object that satisfy a predicate.

$\forall x$ (For all x)

$\exists x$ (At least 1)

Ex→ Not all rainy days are cold

$$\exists_d (\text{Rainy}(d) \wedge \sim(\text{Cold}(d)))$$

at least 1 day Rainy and not cold

Or $\sim [\forall d (R(d) \to C(d))]$

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Algebraic Structures

Group Theory
Study of set of elements present in a group.

Closure Property
$\forall a, b \in A$
then $a * b \in A$
not mult, any operator!

Associative Prop.
$\forall a, b, c \in A$
then $(a*b)*c = a*(b*c)$

Identity Prop
$\forall a \in A$ there must be unique $e \in A$ such that
$a*e = e*a = a$

Algebraic Structure
A non-emty set that satisfies closure prop.
Ex→ $(N, -)$

Semi-Group
A non-empty set A w.r.t binary operation($*$), if A satisfy closure & Associative
Ex→ $(E, \times)$
even

Monoid
Closure + Associative + Identity prop.
Ex→ $(N, \times)$ mult
natural no.

Inverse Prop
$\forall a \in A$ there must be unique element $a^{-1} \in A$ such that
$a*a^{-1} = a^{-1}*a = e$

Group
A non empty set A is a group wi respect to a binary operation $*$, if A satisfy closure, associative, identity & inverse prop. with respect to $*$.

Ex→ $(Z, +)$ $(Z, \times)$
Integers
✗ as 0 doesn't have inverse

Commutative Prop
$\forall a, b \in A$ such that $a*b = b*a$

Abelian Group
Closure + Associative + Identity + Inverse + commutative.
Ex→ $(Z, +)$

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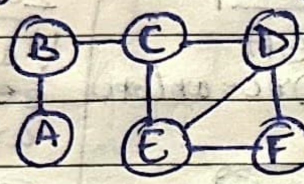
Graph Theory

→ A graph $G(V,E)$ consists of a set of objects $V = \{V_1, V_2 \ldots\}$ called vertices and another set $E = \{E_1, E_2 \ldots\}$ called edges.

→ Each edge e_k is identified with unordered pair (V_i, V_j) of vertices.



2 Types
- Directed
- Undirected

→ Complete / Full Graph

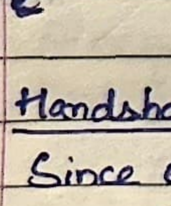
Have edge b\w each & every pair of vertices.

$|V| = n$

$|E| = \frac{n(n-1)}{2}$

Degree of a Vertex

No. of edges associated with a vertex



b = 2
a = 2
d = 1

Handshaking Theorem

Since each edge contribute to 2 degree in graph sum of degree of all vertices in G is twice the no. of edges.

$$\sum_{i=1}^{n} d(V_i) = 2|E|$$

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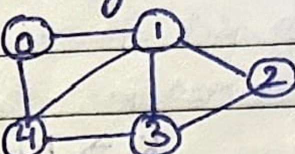
Representation of Graph in memory

→ Adjacency Matrix



	0	1	2	3	4
0	0	1	0	0	1
1	1	0	1	1	1
2	0	1	0	1	0
3	0	1	1	0	1
4	1	1	0	1	0

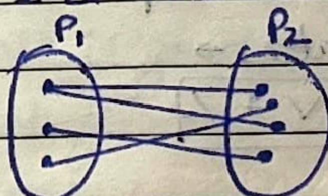
→ Adjacency List

0 → 1 → 4
1 → 0 → 4 → 2
2 → 1 → 3
3 → 1 → 2 → 4
4 → 0 → 1 → 3

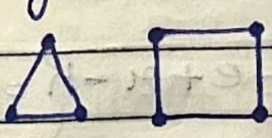
Popular Graphs

→ Bi-partite Graph

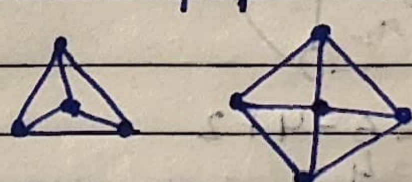
If its vertex set can be partitioned into 2 non-empty disjoint subset such that each edge has 1 end point in group 1 & other in group 2.



→ Cycle Graph



→ Wheel Graph



→ Regular Graph

Every vertex's degree is same.

→ Planer Graph

If a graph can be drawn on a plane in such way that no edges cross each other.

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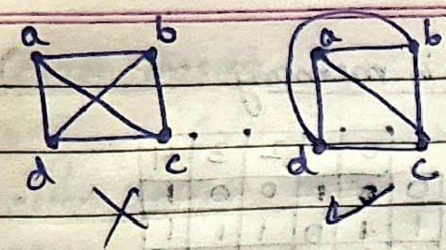
non planar

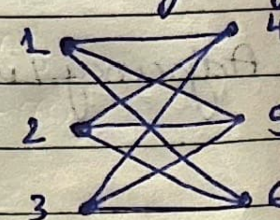


planar

$K_5 \Rightarrow$ Non planar in min no. of vertex

$K_6 \Rightarrow$ Non. planar in min no. of edges

$|V| = 5$
$|E| = 10$



$|V| = 6$
$|E| = 9$

Simplest Non-Planer Graph

1. Kuratowski's case 1 = K_5
2. " " 2 = $K_{3,3}$

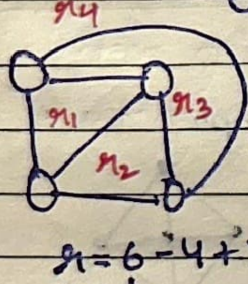
Euler's Formula

A planer graph divides a plane into many regions

Euler's formular states $\rightarrow$

$$\boxed{r = e - v + 2}$$

r → regions

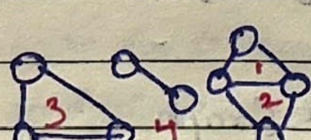


$r = 6 - 4 + 2$
$r = 4$

Euler's Formula on disconnected graph

$$\boxed{v - e + r - K = 1}$$

K → no. of connected components

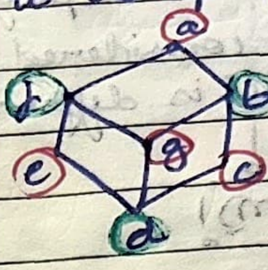


$9 - 9 + r - 3 = 1$
$r = 4$

Graph Coloring
→ Vertex Coloring
→ Region Coloring

Proper Vertex Coloring → Color all vertices such that no 2 adjacent vertex have same color

Chromatic No. of Graph → Min. no. of colors required to do proper vertex coloring.



Chromatic no. = 2

Graph Traversal

Walking the graph.

Open walk → Walking all vertices freely.

Closed walk → Begin & End vertex is same

~~Path → Open walk~~

Euler Graph

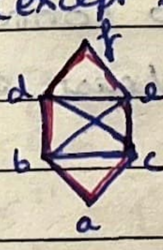
If closed walk in graph contains all edges of a connected graph, then that ~~graph~~ is called Euler graph.

All edges traversed once.

Note → Only possible if all vertices are of even degree

Hamiltonian Graph

Closed walk that traverses every vertex exactly once (except start vertex).



Date / /

Combinatorics

Permutation ~~orders~~

Arrangement of a set of items in a specific order

→ Order Matters : Arrangement is considered different if order of item is different.

→ Counting Permutations: $^{n}P_{m} = \frac{n!}{(n-m)!}$

How it came to be?

Suppose 5 elements

1 2 3 4 5

We have to make arrangement of 3.

$\underline{5} \times \underline{4} \times \underline{3}$ equals $\boxed{^{5}P_{3} = \frac{\lfloor 5}{\lfloor 5-3} = \frac{5\times4\times3\times\cancel{2\times1}}{\cancel{2\times1}}}$

can pick any no. out of 5 to be 1st element

1 choice taken, so still 4 choices left

Combination

Selection of items where order doesn't matter

→ Counting Combination : $^{n}C_{r} = \frac{n!}{(n-r)!\,r!}$

Pigeonhole Principle

If more items are put into few containers then there ~~are items~~ must be atleast 1 container that contains more than 1 item.